# 9 Symmetries and the quark model

Symmetries are a central to our current understanding of particle physics. In this chapter, the concepts of symmetries and conservation laws are first introduced in the general context of quantum mechanics and are then applied to the quark model. The approximate light quark flavour symmetry is used to predict the structure and wavefunctions of the lightest hadronic states. These wavefunctions are used to obtain predictions for the masses and magnetic moments of the observed baryons. The discussion of the quark model provides an introduction to the algebra of the SU(2) and SU(3) symmetry groups that play a central rôle in the Standard Model. No prior knowledge of group theory is assumed; the required properties of the SU(2) and SU(3) symmetry groups are obtained from first principles.

## 9.1 Symmetries in quantum mechanics

In both classical and quantum physics, conservation laws are associated with symmetries of the Hamiltonian. For particle physics it is most natural to introduce these ideas in the context of quantum mechanics. In quantum mechanics, a symmetry of the Universe can be expressed by requiring that all physical predictions are invariant under the wavefunction transformation

$$\psi \to \psi' = \hat{U}\psi,$$

where, for example, $\hat{U}$ could be the operator corresponding a finite rotation of the coordinate axes. The requirement that all physical predictions are unchanged by a symmetry transformation, constrains the possible form of $\hat{U}$. A necessary requirement is that wavefunction normalisations are unchanged, implying

$$\langle\psi|\psi\rangle = \langle\psi'|\psi'\rangle = \langle\hat{U}\psi|\hat{U}\psi\rangle = \langle\psi|\hat{U}^\dagger\hat{U}|\psi\rangle.$$

From this it can be concluded that the operator corresponding to any acceptable symmetry transformation in quantum mechanics must be unitary

$$\boxed{\hat{U}^\dagger\hat{U} = I,}$$

where $I$ represents unity (which could be 1 or the identity matrix). Furthermore, for physical predications to be unchanged by a symmetry operation, the eigenstates of the system also must be unchanged by the transformation. Hence the Hamiltonian itself must possess the symmetry in question, $\hat{H} \to \hat{H}' = \hat{H}$. The eigenstates of the Hamiltonian satisfy

$$\hat{H}\psi_i = E_i\psi,$$

and because of the invariance of the Hamiltonian, the energies of the transformed eigenstates $\psi_i'$ will be unchanged,

$$\hat{H}'\psi_i' = \hat{H}\psi_i' = E_i\psi_i'.$$

Since $\psi_i' = \hat{U}\psi_i$, this implies

$$\hat{H}\hat{U}\psi_i = E_i\hat{U}\psi_i = \hat{U}E_i\psi_i = \hat{U}\hat{H}\psi_i.$$

Therefore, for all states of the system, $\hat{H}\hat{U}\psi_i = \hat{U}\hat{H}\psi_i$, and it can be concluded that $\hat{U}$ commutes with the Hamiltonian

$$\boxed{\left[\hat{H}, \hat{U}\right] \equiv \hat{H}\hat{U} - \hat{U}\hat{H} = 0.}$$

Hence, for each symmetry of the Hamiltonian there is a corresponding unitary operator which commutes with the Hamiltonian.

A finite continuous symmetry operation can be built up from a series of infinitesimal transformations of the form

$$\hat{U}(\epsilon) = I + i\epsilon\hat{G},$$

where $\epsilon$ is an infinitesimally small parameter and $\hat{G}$ is called the generator of the transformation. Since $\hat{U}$ is unitary,

$$\hat{U}(\epsilon)\hat{U}^\dagger(\epsilon) = (I + i\epsilon\hat{G})(I - i\epsilon\hat{G}^\dagger) = I + i\epsilon(\hat{G} - \hat{G}^\dagger) + O(\epsilon^2).$$

For this infinitesimal transformation terms of $O(\epsilon^2)$ can be neglected, and therefore the requirement that $U^\dagger U = I$ implies that

$$\boxed{\hat{G} = \hat{G}^\dagger.}$$

Thus, for each symmetry of the Hamiltonian there is a corresponding unitary symmetry operation with an associated Hermitian generator $\hat{G}$. The eigenstates of a Hermitian operator are real and therefore the operator $\hat{G}$ is associated with an observable quantity $G$. Furthermore, since $\hat{U}$ commutes with the Hamiltonian, $\left[\hat{H}, I + i\epsilon\hat{G}\right] = 0$, the generator $\hat{G}$ also must commute with the Hamiltonian,

$$\left[\hat{H}, \hat{G}\right] = 0.$$

In quantum mechanics the time evolution of the expectation value of the operator $\hat{G}$ is given by (2.29),

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle \hat{G}\rangle = i\left\langle \left[\hat{H}, \hat{G}\right]\right\rangle,$$

and because here $\hat{G}$ commutes with the Hamiltonian,

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle \hat{G}\rangle = 0.$$

Hence, for each symmetry of the Hamiltonian, there is an associated observable *conserved* quantity $G$. Thus in quantum mechanics, symmetries are associated with conservation laws and vice versa. This profound statement is not restricted to quantum mechanics, in classical dynamics symmetries of the Hamiltonian also correspond to conserved quantities. The relationship between symmetries and conservation laws is an expression of Noether's theorem, which associates a symmetry of the Lagrangian with a conserved current (see, for example, Appendix E).

### Translational invariance

As an example of the above arguments, consider the simple case of translational invariance in one dimension. The Hamiltonian for a system of particles depends only on the velocities and the relative distances between particles and therefore does not change if all particles are translated by the same infinitesimal distance $\epsilon$,

$$x \to x + \epsilon.$$

The corresponding wavefunction transformation is

$$\psi(x) \to \psi'(x) = \psi(x+\epsilon).$$

Performing a Taylor expansion of $\psi(x)$ in terms of $\epsilon$ gives

$$\psi'(x) = \psi(x+\epsilon) = \psi(x) + \frac{\partial\psi}{\partial x}\epsilon + O(\epsilon^2).$$

For this infinitesimal transformation, the terms of $O(\epsilon^2)$ can be dropped, giving

$$\psi'(x) = \left(1 + \epsilon\frac{\partial}{\partial x}\right)\psi(x). \tag{9.1}$$

This can be expressed in terms of the quantum-mechanical momentum operator,

$$\hat{p}_x = -i\frac{\partial}{\partial x},$$

giving

$$\psi'(x) = (1 + i\epsilon\hat{p}_x)\psi(x).$$

Comparison with (9.1) shows that the generator of the symmetry transformation, $x \to x + \epsilon$, is the quantum-mechanical momentum operator $\hat{p}_x$. Hence, the translational invariance of Hamiltonian implies momentum conservation.

In general, a symmetry operation may depend on more than one parameter, and the corresponding infinitesimal unitary operator can be written in terms of the set of generators $\hat{\mathbf{G}} = \{\hat{G}_i\}$,

$$\hat{U} = 1 + i\boldsymbol{\epsilon} \cdot \hat{\mathbf{G}},$$

where $\boldsymbol{\epsilon} = \{\epsilon_i\}$. For example, an infinitesimal three-dimensional spatial translation $\mathbf{x} \to \mathbf{x} + \boldsymbol{\epsilon}$ can be associated with the generators $\hat{\mathbf{p}} = (\hat{p}_x, \hat{p}_y, \hat{p}_z)$ with

$$\hat{U}(\boldsymbol{\epsilon}) = 1 + i\boldsymbol{\epsilon} \cdot \hat{\mathbf{p}} \equiv 1 + i\epsilon_x\hat{p}_x + i\epsilon_y\hat{p}_y + \epsilon_z\hat{p}_z.$$

### 9.1.1 Finite transformations

Any finite symmetry transformation can be expressed as a series of infinitesimal transformations using

$$\hat{U}(\boldsymbol{\alpha}) = \lim_{n\to\infty}\left(1 + i\frac{1}{n}\boldsymbol{\alpha} \cdot \hat{\mathbf{G}}\right)^n = \exp\left(i\boldsymbol{\alpha} \cdot \mathbf{G}\right).$$

For example, consider the finite translation $x \to x+x_0$ in one dimension. The corresponding unitary operator, expressed in terms of the generator of the infinitesimal translation $\hat{p}_x$, is

$$\hat{U}(x_0) = \exp\left(ix_0\hat{p}_x\right) = \exp\left(x_0\frac{\partial}{\partial x}\right).$$

Hence for this finite translation, wavefunctions transform according to

$$\begin{aligned}
\psi'(x) = \hat{U}\psi(x) &= \exp\left(x_0\frac{\partial}{\partial x}\right)\psi(x) \\
&= \left(1 + x_0\frac{\partial}{\partial x} + \frac{x_0^2}{2!}\frac{\partial^2}{\partial x^2} + \cdots\right)\psi(x) \\
&= \psi(x) + x_0\frac{\partial\psi}{\partial x} + \frac{x_0^2}{2}\frac{\partial^2\psi}{\partial x^2} + \cdots,
\end{aligned}$$

which is just the usual Taylor expansion for $\psi(x + x_0)$, and therefore $\hat{U}(x_0)$ results in the transformation

$$\psi(x) \to \psi'(x) = \hat{U}(x_0)\psi = \psi(x + x_0),$$

as required.

## 9.2 Flavour symmetry

In the early days of nuclear physics, it was realised that the proton and neutron have very similar masses and that the nuclear force is approximately charge independent. In other words, the strong force potential is the same for two protons, two neutrons or a neutron and a proton

$$V_{pp} \approx V_{np} \approx V_{nn}.$$

Heisenberg suggested that if you could switch off the electric charge of the proton, there would be no way to distinguish between a proton and a neutron. To reflect this observed symmetry of the nuclear force, it was proposed that the neutron and proton could be considered as two states of a single entity, the *nucleon*, analogous to the spin-up and spin-down states of a spin-half particle,

$$\mathrm{p} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathrm{n} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

This led to the introduction of the idea of *isospin*, where the proton and neutron form an isospin doublet with total isospin $I = 1/2$ and third component of isospin $I_3 = \pm 1/2$. The charge independence of the strong nuclear force is then expressed in terms of invariance under unitary transformations in this isospin space. One such transformation would correspond to replacing all protons with neutrons and vice versa. Physically, isospin has nothing to do with spin. Nevertheless, it will be shown in the following section that isospin satisfies the same SU(2) algebra as spin.

### 9.2.1 Flavour symmetry of the strong interaction

The idea of proton/neutron isospin symmetry can be extended to the quarks. Since the QCD interaction treats all quark flavours equally, the strong interaction possesses a flavour symmetry analogous to isospin symmetry of the nuclear force. For a system of quarks, the Hamiltonian can be broken down into three components

$$\hat{H} = \hat{H}_0 + \hat{H}_{\text{strong}} + \hat{H}_{\text{em}}, \tag{9.2}$$

where $\hat{H}_0$ is the kinetic and rest mass energy of the quarks, and $\hat{H}_{\text{strong}}$ and $\hat{H}_{\text{em}}$ are respectively the strong and electromagnetic interaction terms. If the (effective) masses of the up- and down-quarks are the same, and $\hat{H}_{\text{em}}$ is small compared to $\hat{H}_{\text{strong}}$, then to a good approximation the Hamiltonian possesses an up–down (ud) flavour symmetry; nothing would change if all the up-quarks were replaced by down-quarks and vice versa. One simple consequence of an exact ud flavour symmetry is that the existence of a (uud) bound quark state implies that there will a corresponding state (ddu) with the same mass.

The above idea can be developed mathematically by writing the up- and down-quarks as states in an abstract flavour space

$$\mathrm{u} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathrm{d} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

If the up- and down-quarks were indistinguishable, the flavour independence of the QCD interaction could be expressed as an invariance under a general unitary transformation in this abstract space

$$\begin{pmatrix} \mathrm{u}' \\ \mathrm{d}' \end{pmatrix} = \hat{U} \begin{pmatrix} \mathrm{u} \\ \mathrm{d} \end{pmatrix} = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix} \begin{pmatrix} \mathrm{u} \\ \mathrm{d} \end{pmatrix}.$$

Since a general $2 \times 2$ matrix depends on four complex numbers, it can be described by eight real parameters. The condition $\hat{U}\hat{U}^{\dagger} = I$, imposes four constraints; therefore a $2 \times 2$ unitary matrix can be expressed in terms of four real parameters or, equivalently, four linearly independent $2 \times 2$ matrices representing the generators of the transformation

$$\hat{U} = \exp(i\alpha_i \hat{G}_i).$$

One of the generators can be identified as

$$\hat{U} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} e^{i\phi}. \tag{9.3}$$

This $U(1)$ transformation corresponds to multiplication by a complex phase and is therefore not relevant to the discussion of transformations *between* different flavour states. The remaining three unitary matrices form a special unitary SU(2) group with the property[1] $\det U = 1$. The three matrices representing the Hermitian generators of the SU(2) group are linearly independent from the identity and are therefore traceless. A suitable choice[2] for three Hermitian traceless generators of the ud flavour symmetry are the Pauli spin-matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \text{and} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The ud flavour symmetry corresponds to invariance under SU(2) transformations leading to three conserved observable quantities defined by the eigenvalues of Pauli

1 The property $\det U = 1$ follows from the properties of determinants, $\det U^{\dagger}U \equiv \det I = \det U^{\dagger} \det U = \det U^{*} \det U = |\det U|^2 = 1$. For the corresponding infinitesimal transformation to be close to the identity, $\det U$ must equal $+1$.

2 The algebra of the SU(2) is determined by the commutation relations of the generators. The use of the Pauli spin-matrices is purely conventional. An equally valid choice of the generators $G_i$ would be $S^{\dagger}\sigma_i S$ where $S$ is an arbitrary unitary matrix. The commutation relations are unchanged by this redefinition, and thus the algebra of SU(2) does not depend on the specific representation of the generators.

spin-matrices. The *algebra* of the ud flavour symmetry is therefore identical to that of spin for a spin-half particle. In analogy with the quantum-mechanical treatment of spin-half particles, isospin $\hat{\mathbf{T}}$ is defined in terms of the Pauli spin-matrices

$$\boxed{\hat{\mathbf{T}} = \tfrac{1}{2}\boldsymbol{\sigma}.}$$

Any finite transformation in the up–down quark flavour space can be written in terms of a unitary transformation

$$\hat{U} = e^{i\boldsymbol{\alpha}\cdot\hat{\mathbf{T}}},$$

such that

$$\begin{pmatrix} \mathrm{u}' \\ \mathrm{d}' \end{pmatrix} = e^{i\boldsymbol{\alpha}\cdot\hat{\mathbf{T}}} \begin{pmatrix} \mathrm{u} \\ \mathrm{d} \end{pmatrix},$$

where $\boldsymbol{\alpha}\cdot\hat{\mathbf{T}} = \alpha_1\hat{T}_1 + \alpha_2\hat{T}_2 + \alpha_3\hat{T}_3$. Hence, the general flavour transformation is a "rotation" in flavour space, not just the simple interchange of up and down quarks. A general unitary transformation in this isospin space would amount to relabelling the up-quark as a linear combination of the up-quark and the down-quark. If the flavour symmetry were exact, and the up- and down-quarks were genuinely indistinguishable, this would be perfectly acceptable. However, because the up- and down-quarks have different charges, it does not make sense to form states which are linear combinations of the two, as this would lead to violations of electric charge conservation. Consequently, the only physical meaningful isospin transformation is that which corresponds to relabelling the states, u ↔ d.

### 9.2.2 Isospin algebra

Whilst isospin has nothing to do with the physical property of spin, it has exactly the same mathematical structure defined by the generators of the SU(2) symmetry group. In the language of group theory the generators of SU(2) define a non-Abelian (i.e. non-commuting) Lie algebra. The three generators of the group, which correspond to physical observables, satisfy the algebra

$$\left[\hat{T}_1, \hat{T}_2\right] = i\hat{T}_3, \quad \left[\hat{T}_2, \hat{T}_3\right] = i\hat{T}_1 \quad \text{and} \quad \left[\hat{T}_3, \hat{T}_1\right] = i\hat{T}_2.$$

This is exactly the same set of commutators as found for the quantum mechanical treatment of angular momentum, introduced in Section 2.3.5. Consequently, the results obtained for angular momentum can be applied directly to the properties of isospin. The total isospin operator,

$$\hat{T}^2 = \hat{T}_1^2 + \hat{T}_2^2 + \hat{T}_3^2,$$

which commutes with each of the generators, is Hermitian and therefore also corresponds to an observable quantity. Because the three operators $\hat{T}_1$, $\hat{T}_2$ and $\hat{T}_3$ do not

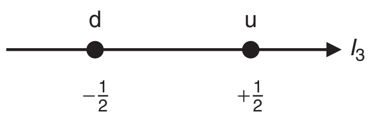


Fig. 9.1 The isospin one-half multiplet consisting of an up-quark and a down-quark.

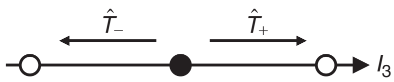


Fig. 9.2 The isospin ladder operators step along the states in $I_3$ within an isospin multiplet.

commute with each other, the corresponding observables cannot be known simultaneously (see Section 2.3.4). Hence, isospin states can be labelled in terms of the total isospin $I$ and the third component of isospin $I_3$. These isospin states $\phi(I, I_3)$ are the mathematical analogues of the angular momentum states $|l, m\rangle$ and have the properties

$$\hat{T}^2\phi(I, I_3) = I(I+1)\phi(I, I_3) \quad \text{and} \quad \hat{T}_3\phi(I, I_3) = I_3\phi(I, I_3).$$

In terms of isospin, the up-quark and down-quark are represented by

$$\text{u} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \phi\left(\tfrac{1}{2}, +\tfrac{1}{2}\right) \quad \text{and} \quad \text{d} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \phi\left(\tfrac{1}{2}, -\tfrac{1}{2}\right).$$

The up- and down-quarks are the two states of an isospin one-half multiplet with respective third components of isospin $+\frac{1}{2}$ and $-\frac{1}{2}$ as indicated in Figure 9.1.

### Isospin ladder operators

The isospin ladder operators, analogous to the quantum mechanical angular momentum ladder operators, defined as

$$\hat{T}_- \equiv \hat{T}_1 - i\hat{T}_2 \quad \text{and} \quad \hat{T}_+ \equiv \hat{T}_1 + i\hat{T}_2,$$

have the effect of moving between the $(2I+1)$ states within an isospin multiplet, as indicated in Figure 9.2. The action the ladder operators on a particular isospin state are

$$\hat{T}_+\phi(I, I_3) = \sqrt{I(I+1) - I_3(I_3+1)}\,\phi(I, I_3+1), \tag{9.4}$$

$$\hat{T}_-\phi(I, I_3) = \sqrt{I(I+1) - I_3(I_3-1)}\,\phi(I, I_3-1), \tag{9.5}$$

where the coefficients were derived in Section 2.3.5. For an isospin multiplet with total isospin $I$, the ladder operators have the effect of raising or lowering the third component of isospin. The action of the ladder operators on the extreme states with $I_3 = \pm I$ yield zero,

$$\hat{T}_-\phi(I, -I) = 0 \quad \text{and} \quad \hat{T}_+\phi(I, +I) = 0.$$

Therefore, the effects of the isospin ladder operators on the u- and d-quarks are

$$\hat{T}_{+}\mathrm{u} = 0, \quad \hat{T}_{+}\mathrm{d} = \mathrm{u}, \quad \hat{T}_{-}\mathrm{u} = \mathrm{d} \quad \text{and} \quad \hat{T}_{-}\mathrm{d} = 0.$$

## 9.3 Combining quarks into baryons

The strong interaction Hamiltonian does not distinguish between up- and down-quarks, therefore in the limit where the up- and down-quark masses are the same, physical predictions involving the strong interaction alone are symmetric under unitary transformations in this space. The conserved observable quantities, corresponding to the generators of this symmetry are $I_3$ and $I$. Because $I_3$ and $I$ are conserved in strong interactions, the concept of isospin is useful in describing low-energy hadron interactions. For example, isospin arguments can be used to explain the observation that the decay rate for $\Delta^+ \to \mathrm{p}\pi^0$ is twice that for $\Delta^0 \to \mathrm{n}\pi^0$ (see Problem 9.3). Here the concept of isospin will be used to construct the flavour wavefunctions of baryons (qqq) and mesons ($\mathrm{q}\overline{\mathrm{q}}$).

The rules for combining isospin for a system of two quarks are identical to those for the addition of angular momentum. The third component of isospin is added as a scalar and the total isospin is added as the magnitude of a vector. If two isospin states $\phi\left(I^a, I_3^a\right)$ and $\phi\left(I^b, I_3^b\right)$ are combined, the resulting isospin state $\phi\left(I, I_3\right)$ has

$$I_3 = I_3^a + I_3^b \quad \text{and} \quad |I^a - I^b| \leq I \leq |I^a + I^b|.$$

These rules can be used to identify the possible isospin states formed from two quarks (each of which can be either an up- or down-quark). The third component of isospin is the scalar sum of $I_3$ for the individual quarks, and hence the $I_3$ assignments of the four possible combinations of two light quarks are those of Figure 9.3. The isospin assignments for the extreme states immediately can be identified as

$$\mathrm{uu} \equiv \phi\left(\tfrac{1}{2}, \tfrac{1}{2}\right)\phi\left(\tfrac{1}{2}, \tfrac{1}{2}\right) = \phi\left(1, +1\right) \quad \text{and} \quad \mathrm{dd} \equiv \phi\left(\tfrac{1}{2}, -\tfrac{1}{2}\right)\phi\left(\tfrac{1}{2}, -\tfrac{1}{2}\right) = \phi\left(1, -1\right).$$

This identification is unambiguous, since a state with $I_3 = \pm 1$ must have $I \geq 1$ and the maximum total isospin for a two-quark state is $I = 1$. The quark combinations ud and du, which both have $I_3 = 0$, are not eigenstates of total isospin. The

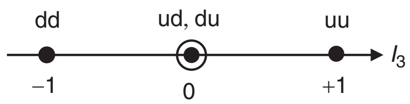


Fig. 9.3 The $I_3$ assignments for the four possible combinations of two up- or down-quarks. There are two states with $I_3 = 0$ (indicated by the point and circle) ud and du.

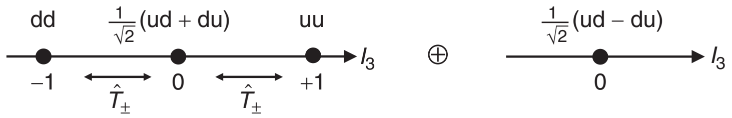


Fig. 9.4 The isospin eigenstates for the combination of two quarks.

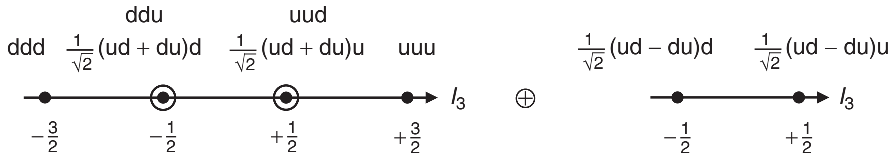


Fig. 9.5 The $I_3$ assignments of three-quark states built from the qq triplet and singlet states.

appropriate linear combination corresponding to the $I = 1$ state can be identified using isospin ladder operators,

$$\hat{T}_-\phi(1,+1) = \sqrt{2}\phi(1,0) = \hat{T}_-(\mathrm{uu}) = \mathrm{ud} + \mathrm{du},$$

and thus

$$\phi(1,0) = \frac{1}{\sqrt{2}}(\mathrm{ud} + \mathrm{du}).$$

The $\phi(0,0)$ state can be identified as the linear combination of ud and du that is orthogonal to $\phi(1,0)$, from which

$$\phi(0,0) = \frac{1}{\sqrt{2}}(\mathrm{ud} - \mathrm{du}). \tag{9.6}$$

Acting on the $I = 0$ singlet state of (9.6) with either $\hat{T}_+$ or $\hat{T}_-$ gives zero, confirming that it is indeed the $\phi(0,0)$ state, for example

$$\begin{aligned}\hat{T}_+\frac{1}{\sqrt{2}}(\mathrm{ud} - \mathrm{du}) &= \frac{1}{\sqrt{2}}\left([\hat{T}_+\mathrm{u}]\mathrm{d} + \mathrm{u}[\hat{T}_+\mathrm{d}] - [\hat{T}_+\mathrm{d}]\mathrm{u} - \mathrm{d}[\hat{T}_+\mathrm{u}]\right)\\ &= \frac{1}{\sqrt{2}}(\mathrm{uu} - \mathrm{uu}) = 0.\end{aligned}$$

The four possible combinations of two isospin *doublets* therefore decomposes into a *triplet* of isospin-1 states and a *singlet* isospin-0 state, as shown in Figure 9.4. This decomposition can be written as $2 \otimes 2 = 3 \oplus 1$. It should be noted that the isospin-0 and isospin-1 states are physically different; the isospin-1 triplet is symmetric under interchange of the two quarks, whereas the isospin singlet is antisymmetric.

The isospin states formed from three quarks can be obtained by adding an up- or down-quark to the qq isospin singlet and triplet states of Figure 9.4. Since $I_3$ adds as a scalar, the $I_3$ assignments of the possible combinations are those shown in Figure 9.5. The two states built from the $I = 0$ singlet will have total isospin $I = 1/2$, whereas those constructed from the $I = 1$ triplet can have either $I = 1/2$ or $I = 3/2$. Of the six combinations formed from the triplet, the extreme ddd and uuu states with $I_3 = -3/2$ and $I_3 = +3/2$ uniquely can be identified as being part of isospin $I = 3/2$ multiplet. The other two $I = 3/2$ states can be identified using the ladder operators. For example, the $\phi\left(\frac{3}{2}, -\frac{1}{2}\right)$ state, which is a linear combination of the ddu and $\frac{1}{\sqrt{2}}$(ud + du)d states, can be obtained from the action of $\hat{T}_+$ on

$$\phi\left(\tfrac{3}{2}, -\tfrac{3}{2}\right) = \mathrm{ddd},$$

from which

$$\begin{aligned}\hat{T}_+\phi\left(\tfrac{3}{2}, -\tfrac{3}{2}\right) = \sqrt{3}\phi\left(\tfrac{3}{2}, -\tfrac{1}{2}\right) &= \hat{T}_+(\mathrm{ddd}) = [\hat{T}_+\mathrm{d}]\mathrm{dd} + \mathrm{d}[\hat{T}_+\mathrm{d}]\mathrm{d} + \mathrm{dd}[\hat{T}_+\mathrm{d}] \\ &= \mathrm{udd} + \mathrm{dud} + \mathrm{ddu},\end{aligned}$$

and therefore

$$\phi\left(\tfrac{3}{2}, -\tfrac{1}{2}\right) = \tfrac{1}{\sqrt{3}}(\mathrm{udd} + \mathrm{dud} + \mathrm{ddu}). \tag{9.7}$$

From the repeated action of the ladder operators, the four isospin-$\frac{3}{2}$ states, built from the qq triplet, can be shown to be

$$\phi\left(\tfrac{3}{2}, -\tfrac{3}{2}\right) = \mathrm{ddd}, \tag{9.8}$$
$$\phi\left(\tfrac{3}{2}, -\tfrac{1}{2}\right) = \tfrac{1}{\sqrt{3}}(\mathrm{udd} + \mathrm{dud} + \mathrm{ddu}), \tag{9.9}$$
$$\phi\left(\tfrac{3}{2}, +\tfrac{1}{2}\right) = \tfrac{1}{\sqrt{3}}(\mathrm{uud} + \mathrm{udu} + \mathrm{duu}), \tag{9.10}$$
$$\phi\left(\tfrac{3}{2}, +\tfrac{3}{2}\right) = \mathrm{uuu}. \tag{9.11}$$

The two states obtained from the qq triplet with total isospin $I = 1/2$ are orthogonal to the $I_3 = \pm 1/2$ states of (9.9) and (9.10). Hence, the $\phi\left(\frac{1}{2}, -\frac{1}{2}\right)$ state can be identified as the linear combination of ddu and $\frac{1}{\sqrt{2}}$(ud + du)d that is orthogonal to the $\phi\left(\frac{3}{2}, -\frac{1}{2}\right)$ state of (9.9), giving

$$\phi_S\left(\tfrac{1}{2}, -\tfrac{1}{2}\right) = -\tfrac{1}{\sqrt{6}}(2\mathrm{ddu} - \mathrm{udd} - \mathrm{dud}), \tag{9.12}$$

and similarly

$$\phi_S\left(\tfrac{1}{2}, +\tfrac{1}{2}\right) = \tfrac{1}{\sqrt{6}}(2\mathrm{uud} - \mathrm{udu} - \mathrm{duu}). \tag{9.13}$$

The relative phases of (9.12) and (9.13) ensure that the ladder operators correctly step between the two states. In addition, the two states constructed from the qq isospin singlet of (9.6) are

$$\phi_A\left(\tfrac{1}{2}, -\tfrac{1}{2}\right) = \tfrac{1}{\sqrt{2}}(\mathrm{udd} - \mathrm{dud}), \tag{9.14}$$
$$\phi_A\left(\tfrac{1}{2}, +\tfrac{1}{2}\right) = \tfrac{1}{\sqrt{2}}(\mathrm{udu} - \mathrm{duu}). \tag{9.15}$$

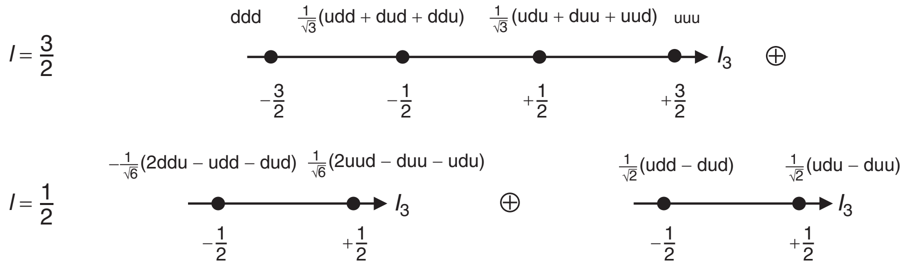


Fig. 9.6 The three-quark $\phi\,(I, I_3)$ states in SU(2) flavour symmetry. The eight combinations decompose into a symmetric quadruplet and two mixed symmetry doublets.

Hence, the eight combinations of three up- and down-quarks, uuu, uud, udu, udd, duu, dud, ddu and ddd, have been grouped into an isospin-$\frac{3}{2}$ *quadruplet* and two isospin-$\frac{1}{2}$ *doublets*, as shown in Figure 9.6. In terms of the SU(2) group structure this can be expressed as

$$2 \otimes 2 \otimes 2 = 2 \otimes (3 \oplus 1) = (2 \otimes 3) \oplus (2 \otimes 1) = 4 \oplus 2 \oplus 2,$$

where $2 \otimes 2 \otimes 2$ represents the combinations of three quarks represented as isospin doublets. The different isospin multiplets have different exchange symmetries. The flavour states in the isospin-$\frac{3}{2}$ quadruplet, (9.8)–(9.11), are symmetric under the interchange of *any* two quarks. The isospin-$\frac{1}{2}$ doublets are referred to as *mixed symmetry* states to reflect the symmetry under the interchange of the first two quarks, but lack of overall exchange symmetry. The doublet states of (9.12) and (9.13), labelled $\phi_S$, are symmetric under the interchange of quarks $1 \leftrightarrow 2$, whereas the doublet states of (9.14) and (9.15), labelled $\phi_A$, are antisymmetric under the interchange of quarks $1 \leftrightarrow 2$. These two isospin doublets have no definite symmetry under the interchange of quarks $1 \leftrightarrow 3$ and $2 \leftrightarrow 3$.

### 9.3.1 Spin states of three quarks

Because the SU(2) algebra for combining spin-half is that same as that for isospin, the possible spin wavefunctions of three quarks, denoted by $\chi$, are constructed in the same manner. Hence the combination of three spin-half particles gives: a spin-$\frac{3}{2}$ quadruplet, with spin states

$$\chi\left(\tfrac{3}{2}, +\tfrac{3}{2}\right) = \uparrow\uparrow\uparrow, \tag{9.16}$$

$$\chi\left(\tfrac{3}{2}, +\tfrac{1}{2}\right) = \tfrac{1}{\sqrt{3}}(\uparrow\uparrow\downarrow + \uparrow\downarrow\uparrow + \downarrow\uparrow\uparrow), \tag{9.17}$$

$$\chi\left(\tfrac{3}{2}, -\tfrac{1}{2}\right) = \tfrac{1}{\sqrt{3}}(\downarrow\downarrow\uparrow + \downarrow\uparrow\downarrow + \uparrow\downarrow\downarrow), \tag{9.18}$$

$$\chi\left(\tfrac{3}{2}, -\tfrac{3}{2}\right) = \downarrow\downarrow\downarrow; \tag{9.19}$$

a mixed symmetry doublet which is symmetric under $1 \leftrightarrow 2$,

$$\chi_S\left(\tfrac{1}{2},-\tfrac{1}{2}\right) = -\tfrac{1}{\sqrt{6}}(2\downarrow\downarrow\uparrow - \uparrow\downarrow\downarrow - \downarrow\uparrow\downarrow), \tag{9.20}$$

$$\chi_S\left(\tfrac{1}{2},+\tfrac{1}{2}\right) = \tfrac{1}{\sqrt{6}}(2\uparrow\uparrow\downarrow - \uparrow\downarrow\uparrow - \downarrow\uparrow\uparrow); \tag{9.21}$$

and a mixed symmetry doublet which is antisymmetric under $1 \leftrightarrow 2$,

$$\chi_A\left(\tfrac{1}{2},-\tfrac{1}{2}\right) = \tfrac{1}{\sqrt{2}}(\uparrow\downarrow\downarrow - \downarrow\uparrow\downarrow), \tag{9.22}$$

$$\chi_A\left(\tfrac{1}{2},+\tfrac{1}{2}\right) = \tfrac{1}{\sqrt{2}}(\uparrow\downarrow\uparrow - \downarrow\uparrow\uparrow). \tag{9.23}$$

## 9.4 Ground state baryon wavefunctions

There are eight possible isospin states for a system of three quarks and eight possible spin states, leading to a total of 64 possible combined flavour and spin states. However, not all combinations satisfy the required fermion exchange symmetry of the total wavefunction. In addition to spin and flavour components, the wavefunction for a qqq state also needs to describe the colour content and the spatial wavefunction. The overall wavefunction for a bound qqq state, accounting for all degrees of freedom, can be written

$$\psi = \phi_{\text{flavour}}\,\chi_{\text{spin}}\,\xi_{\text{colour}}\,\eta_{\text{space}}. \tag{9.24}$$

Because quarks are fermions, the overall wavefunction of (9.24) is required to be antisymmetric under the interchange of *any* two of the quarks. For a system of like fermions, for example uuu, this is simply a statement of the Pauli exclusion principle. However, because of the assumed SU(2) flavour symmetry, when the flavour wavefunction is included, the fermion exchange symmetry applies to the wavefunction as a whole (the argument is given in the starred Addendum in Section 9.7 at the end of this chapter).

The requirement that the wavefunction of (9.24) is totally antisymmetric places restrictions on the individual parts. In Chapter 10, it is shown that the colour wavefunction is necessarily totally antisymmetric. Here the discussion is restricted to the $L = 0$ ground state baryons, in which there is no orbital angular momentum. In this case, the quarks are described by $\ell = 0$ $s$-waves. Since the exchange symmetry of the orbital states is given by $(-1)^\ell$, here the orbital wavefunction is symmetric under the interchange of any two quarks. Consequently, for the $L = 0$ baryons the combination $\xi_{\text{colour}}\,\eta_{\text{space}}$ is antisymmetric under the interchange of any two quarks. For the overall wavefunction to be antisymmetric, the combined flavour and spin wavefunctions, $\phi_{\text{flavour}}\,\chi_{\text{spin}}$, must be symmetric.

$$L = 0 \text{ baryons:} \quad \phi_{\text{flavour}}\,\chi_{\text{spin}} = \text{symmetric}.$$

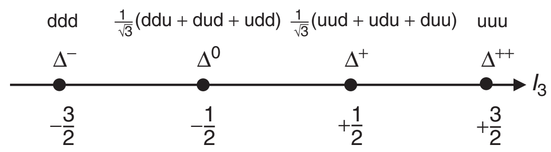


Fig. 9.7 The flavour wavefunctions of the $I = \frac{3}{2}$ light quark $\Delta$-baryons, each of which has total spin $s = \frac{3}{2}$.

The possible forms of the flavour and spin parts of the wavefunction are respectively given by (9.8)–(9.15) and (9.16)–(9.23). There are two ways to construct a totally symmetric combination of $\phi_{\text{flavour}}$ and $\chi_{\text{spin}}$. Firstly, the totally symmetric flavour wavefunctions of (9.8)–(9.11) can be combined with the totally symmetric spin wavefunctions of (9.16)–(9.19) to give four spin-$\frac{3}{2}$, isospin-$\frac{3}{2}$ baryons. These are known as the $\Delta$-baryons with the flavour wavefunctions shown in Figure 9.7.

The second way to construct a totally symmetric $\phi_{\text{flavour}}\chi_{\text{spin}}$ wavefunction is to note that the combinations of mixed symmetry wavefunctions, $\phi_S\chi_S$ and $\phi_A\chi_A$, are both symmetric under interchange of quarks $1 \leftrightarrow 2$. However, neither combination on its own has a definite symmetry under the interchange of quarks $1 \leftrightarrow 3$ and $2 \leftrightarrow 3$. Nevertheless, it is easy to verify that the linear combination

$$\psi = \frac{1}{\sqrt{2}}(\phi_S\chi_S + \phi_A\chi_A) \tag{9.25}$$

is symmetric under the interchange of any two quarks, as required. Here the two possible flavour states correspond to the spin-half proton (uud) and neutron (ddu). Therefore, from (9.25), the wavefunction for a spin-up proton can be identified as

$$\begin{aligned}|\mathrm{p}\uparrow\rangle &= \frac{1}{\sqrt{2}}\left[\phi_S\left(\tfrac{1}{2},+\tfrac{1}{2}\right)\chi_S\left(\tfrac{1}{2},+\tfrac{1}{2}\right)+\phi_A\left(\tfrac{1}{2},+\tfrac{1}{2}\right)\chi_A\left(\tfrac{1}{2},+\tfrac{1}{2}\right)\right]\\ &= \frac{1}{6\sqrt{2}}(2\mathrm{uud}-\mathrm{udu}-\mathrm{duu})(2\uparrow\uparrow\downarrow-\uparrow\downarrow\uparrow-\downarrow\uparrow\uparrow)+\frac{1}{2\sqrt{2}}(\mathrm{udu}-\mathrm{duu})(\uparrow\downarrow\uparrow-\downarrow\uparrow\uparrow),\end{aligned}$$

which when written out in full is

$$\boxed{\begin{aligned}|\mathrm{p}\uparrow\rangle = \frac{1}{\sqrt{18}}(&2\mathrm{u}\uparrow\mathrm{u}\uparrow\mathrm{d}\downarrow-\mathrm{u}\uparrow\mathrm{u}\downarrow\mathrm{d}\uparrow-\mathrm{u}\downarrow\mathrm{u}\uparrow\mathrm{d}\uparrow\\ &+2\mathrm{u}\uparrow\mathrm{d}\downarrow\mathrm{u}\uparrow-\mathrm{u}\uparrow\mathrm{d}\uparrow\mathrm{u}\downarrow-\mathrm{u}\downarrow\mathrm{d}\uparrow\mathrm{u}\uparrow\\ &+2\mathrm{d}\downarrow\mathrm{u}\uparrow\mathrm{u}\uparrow-\mathrm{d}\uparrow\mathrm{u}\uparrow\mathrm{u}\downarrow-\mathrm{d}\uparrow\mathrm{u}\downarrow\mathrm{u}\uparrow).\end{aligned}} \tag{9.26}$$

The fully antisymmetric version of the proton wavefunction would include the antisymmetric colour wavefunction, which itself has six terms, giving a wavefunction with a total of 54 terms with different combinations of flavour, spin and colour. In practice, the wavefunction of (9.26) is sufficient to calculate the physical properties of the proton.

## 9.5 Isospin representation of antiquarks

In the above description of SU(2) flavour symmetry, the up- and down-quarks were placed in an isospin doublet,

$$\mathrm{q} = \begin{pmatrix} \mathrm{u} \\ \mathrm{d} \end{pmatrix}.$$

A general SU(2) transformation of the quark doublet, $\mathrm{q} \to \mathrm{q}' = U\mathrm{q}$, can be written

$$\begin{pmatrix} \mathrm{u} \\ \mathrm{d} \end{pmatrix} \to \begin{pmatrix} \mathrm{u}' \\ \mathrm{d}' \end{pmatrix} = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix} \begin{pmatrix} \mathrm{u} \\ \mathrm{d} \end{pmatrix}, \tag{9.27}$$

where $a$ and $b$ are complex numbers which satisfy $aa^* + bb^* = 1$. In Section 4.7.5, the charge conjugation operation was identified as $\psi' = \hat{C}\psi = i\gamma^2\psi^*$. Hence taking the complex conjugate of (9.27) gives the transformation properties of the flavour part of the antiquark wavefunctions

$$\begin{pmatrix} \bar{\mathrm{u}}' \\ \bar{\mathrm{d}}' \end{pmatrix} = U^* \begin{pmatrix} \bar{\mathrm{u}} \\ \bar{\mathrm{d}} \end{pmatrix} = \begin{pmatrix} a^* & b^* \\ -b & a \end{pmatrix} \begin{pmatrix} \bar{\mathrm{u}} \\ \bar{\mathrm{d}} \end{pmatrix}. \tag{9.28}$$

In SU(2) it is possible to place the antiquarks in a doublet that transforms in the same way as the quarks, $\bar{\mathrm{q}} \to \bar{\mathrm{q}}' = U\bar{\mathrm{q}}$. If the antiquark doublet is written as

$$\bar{\mathrm{q}} \equiv \begin{pmatrix} -\bar{\mathrm{d}} \\ \bar{\mathrm{u}} \end{pmatrix} = S \begin{pmatrix} \bar{\mathrm{u}} \\ \bar{\mathrm{d}} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \bar{\mathrm{u}} \\ \bar{\mathrm{d}} \end{pmatrix}, \tag{9.29}$$

then since

$$\begin{pmatrix} \bar{\mathrm{u}} \\ \bar{\mathrm{d}} \end{pmatrix} = S^{-1}\bar{\mathrm{q}} \quad \text{and} \quad \begin{pmatrix} \bar{\mathrm{u}}' \\ \bar{\mathrm{d}}' \end{pmatrix} = S^{-1}\bar{\mathrm{q}}',$$

Equation (9.28) can be written

$$\begin{aligned} S^{-1}\bar{\mathrm{q}}' &= U^* S^{-1}\bar{\mathrm{q}} \\ \Rightarrow \qquad \bar{\mathrm{q}}' &= S\,U^* S^{-1}\bar{\mathrm{q}}. \end{aligned}$$

Using the definition of the $S$ of (9.29),

$$S\,U^* S^{-1} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a^* & b^* \\ -b & a \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix} = U,$$

and therefore, as desired,

$$\bar{\mathrm{q}} \to \bar{\mathrm{q}}' = U\bar{\mathrm{q}}.$$


Fig. 9.8 The isospin representation of d and u quarks and $\bar{d}$ and $\bar{u}$ antiquarks.

Hence, by placing the antiquarks in an SU(2) doublet defined by

$$\bar{q} \equiv \begin{pmatrix} -\bar{d} \\ \bar{u} \end{pmatrix},$$

the antiquarks transform in exactly the same manner as the quarks. The ordering of the $\bar{d}$ and $\bar{u}$ in the doublet and the minus sign in front of the $\bar{d}$, ensure that quarks and antiquarks behave in the same way under SU(2) flavour transformations and that physical predictions are invariant under the simultaneous transformations of the form $u \leftrightarrow d$ and $\bar{u} \leftrightarrow \bar{d}$. The $I_3$ assignments of the quark and antiquark doublets are shown in Figure 9.8. The effect of the isospin ladder operators on the antiquark doublet can be seen to be

$$T_+\bar{u} = -\bar{d}, \quad T_+\bar{d} = 0, \quad T_-\bar{u} = 0 \quad \text{and} \quad T_-\bar{d} = -\bar{u}.$$

It is important to note that, in general, it is not possible to place the quarks and antiquarks in the same representation; this is a feature SU(2). It *cannot* be applied to the SU(3) flavour symmetry of Section 9.6.

### Meson states

A meson is a bound state of a quark and an antiquark. In terms of isospin, the four possible states formed from up- and down-quarks/antiquarks can be expressed as the combination of an SU(2) quark doublet and an SU(2) antiquark doublet. Using the isospin assignments of Figure 9.8, the $d\bar{u}$ state immediately can be identified as the $q\bar{q}$ isospin state, $\phi(1,-1)$. The two other members of the isospin triplet can be identified by application of the isospin ladder operator $\hat{T}_+$ leading to

$$\begin{aligned}
\phi(1,-1) &= d\bar{u}, \\
\phi(1,0) &= \tfrac{1}{\sqrt{2}}(u\bar{u} - d\bar{d}), \\
\phi(1,+1) &= -u\bar{d}.
\end{aligned}$$

The isospin singlet, which must be orthogonal to the $\phi(1,0)$ state, is therefore

$$\phi(0,0) = \tfrac{1}{\sqrt{2}}\left(u\bar{u} + d\bar{d}\right).$$

This decomposition into an isospin triplet and an isospin singlet, shown in Figure 9.9, is expressed as $2 \otimes \overline{2} = 3 \oplus 1$, where the 2 is the isospin representation of the quark doublet and the $\overline{2}$ is the isospin representation of an antiquark

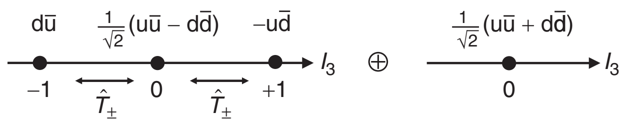


Fig. 9.9 The $q\bar{q}$ isospin triplet and singlet states.

doublet (in the language of group theory the quark doublet is a fundamental representation of SU(2) and the antiquark doublet is the conjugate representation). The action of the isospin raising and lowering operators on the $\phi(0,0)$ state both give zero, confirming that it is indeed a singlet state.

## 9.6 SU(3) flavour symmetry

The SU(2) flavour symmetry described above is almost exact because the difference in the masses of the up- and down-quarks is small and the Coulomb interaction represents a relatively small contribution to the overall Hamiltonian compared to the strong interaction. It is possible to extend the flavour symmetry to include the strange quark. The strong interaction part of the Hamiltonian of (9.2) treats all quarks equally and therefore possesses an exact uds flavour symmetry. However, since the mass of the strange quark is different from the masses of the up- and down-quarks, the overall Hamiltonian is not flavour symmetric. Nevertheless, the difference between $m_s$ and $m_{u/d}$, which is of the order 100 MeV, is relatively small compared to the typical binding energies of baryons, which are of order 1 GeV. It is therefore possible to proceed as if the overall Hamiltonian possessed a uds flavour symmetry. However, the results based on this assumption should be treated with care as, in reality, the symmetry is only approximate.

The assumed uds flavour symmetry can be expressed by a unitary transformation in flavour space

$$\begin{pmatrix} \mathrm{u}' \\ \mathrm{d}' \\ \mathrm{s}' \end{pmatrix} = \hat{U} \begin{pmatrix} \mathrm{u} \\ \mathrm{d} \\ \mathrm{s} \end{pmatrix} = \begin{pmatrix} U_{11} & U_{12} & U_{13} \\ U_{21} & U_{22} & U_{23} \\ U_{31} & U_{32} & U_{33} \end{pmatrix} \begin{pmatrix} \mathrm{u} \\ \mathrm{d} \\ \mathrm{s} \end{pmatrix}.$$

In general, a $3 \times 3$ matrix can be written in terms of nine complex numbers, or equivalently 18 real parameters. There are nine constraints from requirement of unitarity, $\hat{U}^\dagger \hat{U} = I$. Therefore $\hat{U}$ can be expressed in terms of nine linearly independent $3 \times 3$ matrices. As before, one of these matrices is the identity matrix multiplied by a complex phase and is not relevant to the discussion of transformations between different flavour states. The remaining eight matrices form an SU(3) group and can be expressed in terms of the eight independent Hermitian generators $\hat{T}_i$ such that the general SU(3) flavour transformation can be expressed as

$$\hat{U} = e^{i\boldsymbol{\alpha}\cdot\hat{\mathbf{T}}}.$$

The eight generators are written in terms of eight $\lambda$-matrices with

$$\hat{\mathbf{T}} = \frac{1}{2}\boldsymbol{\lambda},$$

where the matrices act on the SU(3) representations of the u, d and s quarks

$$\mathrm{u} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathrm{d} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathrm{s} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \tag{9.30}$$

The SU(3) uds flavour symmetry contains the subgroup of SU(2) u $\leftrightarrow$ d flavour symmetry. Hence, three of the $\lambda$-matrices correspond to the SU(2) ud isospin symmetry and have the Pauli spin-matrices in the top left $2\times2$ block of the $3\times3$ matrix with all other entries zero,

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The third component of isospin is now written in terms of the operator

$$\hat{T}_3 = \tfrac{1}{2}\lambda_3,$$

such that

$$\hat{T}_3\mathrm{u} = +\tfrac{1}{2}\mathrm{u}, \quad \hat{T}_3\mathrm{d} = -\tfrac{1}{2}\mathrm{d} \quad \text{and} \quad \hat{T}_3\mathrm{s} = 0.$$

As before, isospin lowering and raising operators are defined as $T_\pm = \frac{1}{2}(\lambda_1 \pm i\lambda_2)$.

The remaining $\lambda$-matrices can be identified by realising that the SU(3) uds flavour symmetry also contains the subgroups of SU(2) u $\leftrightarrow$ s and SU(2) d $\leftrightarrow$ s flavour symmetries, both of which can also be expressed in terms of the Pauli spin-matrices. The corresponding $3 \times 3$ $\lambda$-matrixes for the u $\leftrightarrow$ s symmetry are

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \quad \text{and} \quad \lambda_X = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

and for the d $\leftrightarrow$ s symmetry they are

$$\lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \quad \text{and} \quad \lambda_Y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Of the nine $\lambda$-matrices identified above, only eight are independent; one of the three diagonal matrices, $\lambda_3$, $\lambda_X$ and $\lambda_Y$, can be expressed in terms of the other two. Because the u $\leftrightarrow$ d symmetry is nearly exact, it is natural to retain $\lambda_3$ as one of the eight generators of the SU(3) flavour symmetry. The final generator is chosen as the linear combination of $\lambda_X$ and $\lambda_Y$ that treats u and d quarks symmetrically

$$\lambda_8 = \frac{1}{\sqrt{3}}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} + \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

The eight matrices used to represent the generators of the SU(3) symmetry, known as the Gell-Mann matrices, are therefore

$$\begin{gathered}
\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \\
\lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \\
\lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_8 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.
\end{gathered} \tag{9.31}$$

### 9.6.1 SU(3) flavour states

For the case of SU(2) flavour symmetry there are three Hermitian generators, each of which corresponds to an observable quantity. However, since the generators do not commute, they correspond to a set of incompatible variables. Consequently SU(2) states were defined in terms of the eigenstates of the third component of isospin $\hat{T}_3$ and the total isospin $\hat{T}^2 = \hat{T}_1^2 + \hat{T}_2^2 + \hat{T}_3^2$. In SU(3) there is an analogue of total isospin, which for the fundamental representation of the quarks can be written

$$\hat{T}^2 = \sum_{i=1}^{8} \hat{T}_i^2 = \frac{1}{4}\sum_{i=1}^{8} \lambda_i^2 = \frac{4}{3}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Of the eight SU(3) generators, only $T_3 = \frac{1}{2}\lambda_3$ and $T_8 = \frac{1}{2}\lambda_8$ commute and therefore describe compatible observable quantities. Hence, in addition to the analogue of the total isospin, SU(3) states are described in terms of the eigenstates of the $\lambda_3$ and $\lambda_8$ matrices. The corresponding quantum numbers are the third component of isospin and the flavour *hypercharge* defined by the operators

$$\hat{T}_3 = \frac{1}{2}\lambda_3 \quad \text{and} \quad \hat{Y} = \frac{1}{\sqrt{3}}\lambda_8.$$

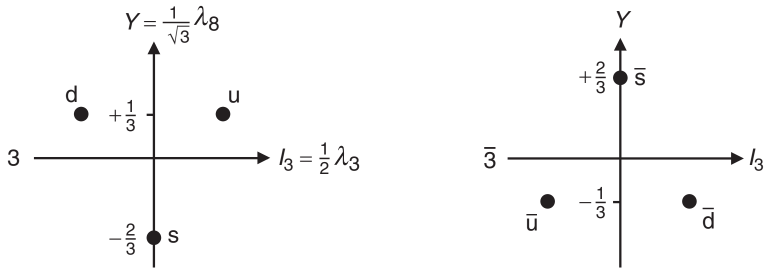


Fig. 9.10 Isospin and hypercharge in SU(3) flavour symmetry for the quarks and antiquarks.

The quarks are the fundamental "3" representation of the SU(3) flavour symmetry. Using the definitions of the quark states of (9.30) it is easy to verify that the isospin and hypercharge assignments of the u, d and s quarks are

$$\hat{T}_3 \mathrm{u} = +\tfrac{1}{2}\mathrm{u} \quad \text{and} \quad \hat{Y}\mathrm{u} = +\tfrac{1}{3}\mathrm{u},$$
$$\hat{T}_3 \mathrm{d} = -\tfrac{1}{2}\mathrm{d} \quad \text{and} \quad \hat{Y}\mathrm{d} = +\tfrac{1}{3}\mathrm{d},$$
$$\hat{T}_3 \mathrm{s} = 0 \quad \text{and} \quad \hat{Y}\mathrm{s} = -\tfrac{2}{3}\mathrm{s}.$$

The flavour content of a state is uniquely identified by $I_3 = n_\mathrm{u} - n_\mathrm{d}$ and $Y = \frac{1}{3}(n_\mathrm{u} + n_\mathrm{d} - 2n_\mathrm{s})$, where $n_\mathrm{u}$, $n_\mathrm{d}$ and $n_\mathrm{s}$ are the respective numbers of up-, down- and strange quarks. The $I_3$ and $Y$ quantum numbers of the antiquarks have the opposite signs compared to the quarks and they form a $\overline{3}$ multiplet, as shown in Figure 9.10.

Whilst the Gell-Mann $\lambda_3$ and $\lambda_8$ matrices label the SU(3) states, the six remaining $\lambda$-matrices can be used to define ladder operators,

$$\hat{T}_\pm = \tfrac{1}{2}(\lambda_1 \pm i\lambda_2),$$
$$\hat{V}_\pm = \tfrac{1}{2}(\lambda_4 \pm i\lambda_5),$$
$$\hat{U}_\pm = \tfrac{1}{2}(\lambda_6 \pm i\lambda_7),$$

which respectively step along the d $\leftrightarrow$ u, s $\leftrightarrow$ u and d $\leftrightarrow$ s directions. From the matrix representations of these ladder operators it is straightforward to verify that

$$\hat{V}_+\mathrm{s} = +\mathrm{u},\ \ \hat{V}_-\mathrm{u} = +\mathrm{s},\ \ \hat{U}_+\mathrm{s} = +\mathrm{d},\ \ \hat{U}_-\mathrm{d} = +\mathrm{s},\ \ \hat{T}_+\mathrm{d} = +\mathrm{u}\ \text{ and }\ \hat{T}_-\mathrm{u} = +\mathrm{d},$$

with all other combinations giving zero. In SU(3) flavour symmetry it is not possible to express the antiquarks as a triplet which transforms in the same way as the quark triplet. Nevertheless, following the arguments given in Section 9.5, the effect of a single ladder operator on an antiquark state must reproduce that from the corresponding SU(2) subgroup, such that the states can be obtained from

$$\hat{V}_+\overline{\mathrm{u}} = -\overline{\mathrm{s}},\ \ \hat{V}_-\overline{\mathrm{s}} = -\overline{\mathrm{u}},\ \ \hat{U}_+\overline{\mathrm{d}} = -\overline{\mathrm{s}},\ \ \hat{U}_-\overline{\mathrm{s}} = -\overline{\mathrm{d}},\ \ \hat{T}_+\overline{\mathrm{u}} = -\overline{\mathrm{d}}\ \text{ and }\ \hat{T}_-\overline{\mathrm{d}} = -\overline{\mathrm{u}}.$$

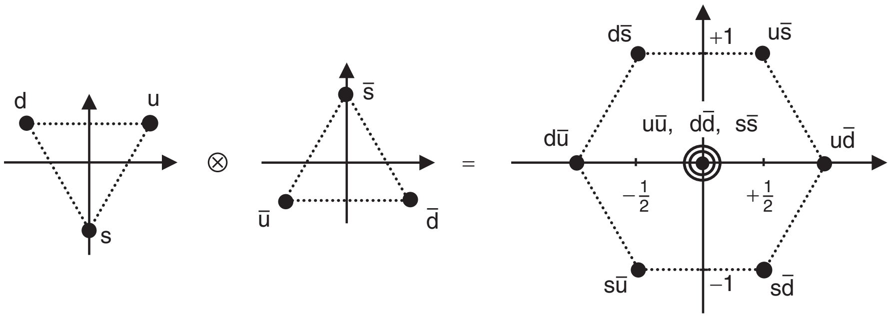


Fig. 9.11 SU(3) isospin and hypercharge assignments of the nine possible $q\bar{q}$ combinations.

### 9.6.2 The light mesons

In the discussion of SU(2) flavour symmetry, the third component of isospin is an additive quantum number, in analogy with angular momentum. In SU(3) flavour symmetry, both $I_3$ and $Y$ are additive quantum numbers, which together specify the flavour content of a state. The light meson ($q\bar{q}$) states, formed from combinations of u, d and s quarks/antiquarks, can be constructed using this additive property to identify the extreme states within an SU(3) multiplet. Having identified the extreme states, the ladder operators can be used to obtain the full multiplet structure. The $I_3$ and $Y$ values for all nine possible combinations of a light quark and a light antiquark are shown in Figure 9.11. The pattern of states can be obtained quickly by drawing triangles corresponding to the antiquark multiplet centred on each of the three positions in the original quark multiplet (this is equivalent to adding the $I_3$ and $Y$ values for all nine combinations).

The states around the edge of the multiplet are uniquely defined in terms of their flavour content. The three physical states with $I_3 = Y = 0$ will be linear combinations of $u\bar{u}$, $d\bar{d}$ and $s\bar{s}$, however, they are not necessarily part of the same multiplet. The $I_3 = Y = 0$ states which are in the same multiplet as the $\{u\bar{s}, u\bar{d}, d\bar{u}, d\bar{s}, s\bar{u}, s\bar{d}\}$ states can be obtained using the ladder operators, as indicated in Figure 9.12,

$$T_+|d\bar{u}\rangle = |u\bar{u}\rangle - |d\bar{d}\rangle \quad \text{and} \quad T_-|u\bar{d}\rangle = |d\bar{d}\rangle - |u\bar{u}\rangle, \tag{9.32}$$

$$V_+|s\bar{u}\rangle = |u\bar{u}\rangle - |s\bar{s}\rangle \quad \text{and} \quad V_-|u\bar{s}\rangle = |s\bar{s}\rangle - |u\bar{u}\rangle, \tag{9.33}$$

$$U_+|s\bar{d}\rangle = |d\bar{d}\rangle - |s\bar{s}\rangle \quad \text{and} \quad U_-|d\bar{s}\rangle = |s\bar{s}\rangle - |d\bar{d}\rangle. \tag{9.34}$$

Of these six states, only two are linearly independent and therefore, of the three physical $I_3 = Y = 0$ states, it can be concluded that one must be in a different SU(3) multiplet. Hence, for the assumed SU(3) flavour symmetry, the $q\bar{q}$ flavour states are decomposed into an *octet* and a *singlet*. The singlet state $\psi_S$ is the linear

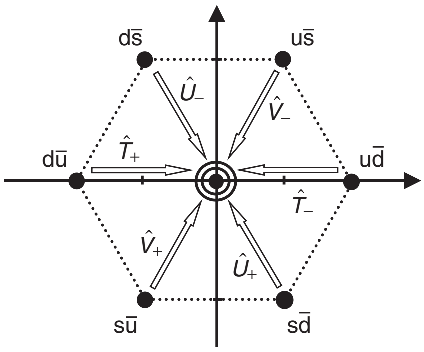


Fig. 9.12 Ladder operators applied to the $q\bar{q}$ states around the edge of the $I_3$, $Y$ diagram.

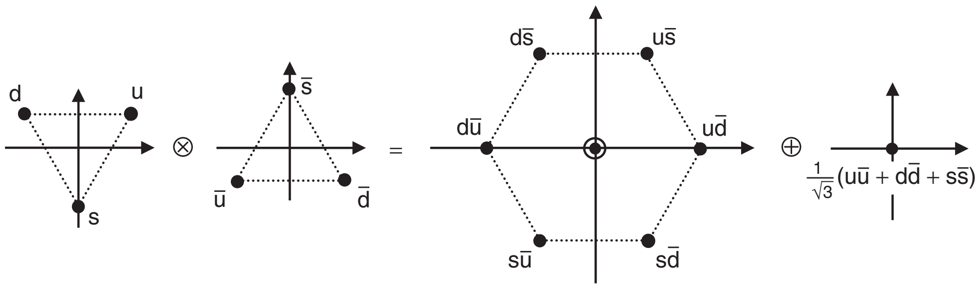


Fig. 9.13 SU(3) flavour $q\bar{q}$ multiplets. The two states at the centre of the octet are linear combinations of $|u\bar{u}\rangle$, $|d\bar{d}\rangle$ and $|s\bar{s}\rangle$ which are orthogonal to the singlet state.

combination of $u\bar{u}$, $d\bar{d}$ and $s\bar{s}$ that is orthogonal to the states of (9.32)–(9.34) and is readily identified as

$$|\psi_S\rangle = \frac{1}{\sqrt{3}}(u\bar{u} + d\bar{d} + s\bar{s}). \tag{9.35}$$

The application of the SU(3) ladder operators on $|\psi_S\rangle$ all give zero, for example

$$\begin{aligned} T_+\psi_S &= \frac{1}{\sqrt{3}}([T_+u]\bar{u} + u[T_+\bar{u}] + [T_+d]\bar{d} + d[T_+\bar{d}] + [T_+s]\bar{s} + s[T_+\bar{s}]) \\ &= \frac{1}{\sqrt{3}}(0 - u\bar{d} + u\bar{d} + 0 + 0 + 0) = 0, \end{aligned}$$

confirming that $|\psi_S\rangle$ is the singlet state.

Figure 9.13 shows the multiplet structure for combining a quark and an antiquark in SU(3) flavour symmetry. In the language of group theory, the combination of a quark 3 representation and an antiquark $\bar{3}$ representation decomposes into an octet and a singlet, $3 \otimes \bar{3} = 8 \oplus 1$. It worth pausing to consider the physical significance of the singlet state. For spin, the corresponding singlet state for the combination of two spin-half states, $|s, m\rangle = |0, 0\rangle$, is a state of zero angular momentum that carries

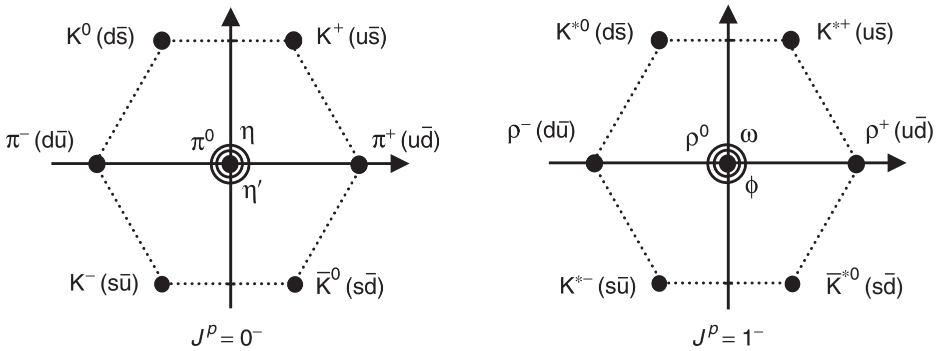


**Fig. 9.14** The nine $\ell = 0, s = 0$ pseudoscalar mesons and nine $\ell = 0, s = 1$ vector mesons formed from the light quarks, plotted in terms of $I_3$ and $Y$.

no information about the spins of its constituent particles; it could just have been formed from two scalar particles. Similarly, the SU(3) flavour singlet $|\psi_S\rangle$ can be thought of as a "flavourless" state, carrying no information about the flavours of its constituents.

### The $L = 0$ mesons

In general, the wavefunction for a meson can be written in terms of four components,

$$\psi(\text{meson}) = \phi_{\text{flavour}}\,\chi_{\text{spin}}\,\xi_{\text{colour}}\,\eta_{\text{space}}.$$

Because quarks and antiquarks are distinguishable, there is no restriction on the exchange symmetry of the wavefunction for a $q\overline{q}$ state. For each flavour state, there are two possible spin states, $s = 0$ and $s = 1$. For the lightest mesons, which have zero orbital angular momentum ($\ell = 0$), the total angular momentum $\mathbf{J}$ is determined by the spin state alone. Consequently the lightest mesons divide into the $J = 0$ pseudoscalar mesons and the $J = 1$ (the vector mesons), respectively with $s = 0$ and $s = 1$. Since quarks and antiquarks have opposite *intrinsic* parities, the overall parity is given by

$$P(q\overline{q}) = P(q)P(\overline{q}) \times (-1)^{\ell} = (+1)(-1)(-1)^{\ell},$$

where $(-1)^{\ell}$ is the symmetry of the orbital wavefunction. Hence, the lightest mesons (with $\ell = 0$) have odd intrinsic parities. In Chapter 10, it is shown that there is only one possible colour wavefunction for a bound $q\overline{q}$ system. Therefore, there are nine light $J^P = 0^-$ pseudoscalar mesons and nine $J^P = 1^-$ light vector mesons, corresponding to nine possible flavour states each with two possible spin states.

Figure 9.14 shows the observed $\ell = 0$ meson states plotted in terms of $I_3$ and $Y$. The $\pi^0$, $\eta$ and $\eta'$ can be associated with the two $I_3 = Y = 0$ octet states and the

$I_3 = Y = 0$ singlet state of Figure 9.13. The $\eta'$, which has an anomalously large mass, can be identified as the singlet state with wavefunction

$$|\eta'\rangle \approx \frac{1}{\sqrt{3}}(u\bar{u} + d\bar{d} + s\bar{s}).$$

If the SU(3) flavour symmetry were exact, the two $I_3 = Y = 0$ octet states would have exactly the same mass and the flavour wavefunctions could be taken to be any two orthogonal linear combinations of (9.32)–(9.34). However, because $m_s > m_{u/d}$, the SU(3) flavour symmetry is only approximate and the choice of the flavour wavefunctions for the observed states will lead to different physical predictions. Experimentally, the lightest pseudoscalar mesons, namely the $\pi^+$, $\pi^0$ and $\pi^-$, are observed to have approximately the same mass of about 140 MeV. Since the $\pi^+$ and $\pi^-$ correspond to the $u\bar{d}$ and $d\bar{u}$ states, the $\pi^0$ can be identified as

$$|\pi^0\rangle = \frac{1}{\sqrt{2}}(u\bar{u} - d\bar{d}).$$

The final $I_3 = Y = 0$ pseudoscalar meson, the $\eta$, is the linear combination of $u\bar{u}$, $d\bar{d}$ and $s\bar{s}$ that is orthogonal to both the $|\eta'\rangle$ and the $|\pi^0\rangle$ states,

$$|\eta\rangle = \frac{1}{\sqrt{6}}(u\bar{u} + d\bar{d} - 2s\bar{s}).$$

In the case of the vector mesons, the predictions of the SU(3) flavour symmetry prove to be less useful; the physical $I_3 = Y = 0$ states are mixtures of the octet and singlet states. Experimentally, the observed states are found to correspond to

$$|\rho^0\rangle = \frac{1}{\sqrt{2}}(u\bar{u} - d\bar{d}),$$
$$|\omega\rangle \approx \frac{1}{\sqrt{2}}(u\bar{u} + d\bar{d}),$$
$$|\phi\rangle \approx s\bar{s}.$$

### 9.6.3 Meson masses

The measured masses of the $\ell = 0$ pseudoscalar and vector mesons are listed in Table 9.1. If the SU(3) flavour symmetry were exact, all the states in pseudoscalar meson octet would have the same mass. The observed mass differences can be ascribed to the fact that the strange quark is more massive than the up- and down-quarks. However, this does not explain why the vector mesons are more massive than their pseudoscalar counterparts. For example, the flavour wavefunctions for the $\pi$ and the $\rho$ states are the same, but their masses are very different. The only difference between the pseudoscalar and vector mesons is the spin wavefunction. Therefore, the different masses of the $\pi$ and $\rho$ mesons can be attributed to a spin–spin interaction.

**Table 9.1** The $L = 0$ pseudoscalar and vector meson masses.

| Pseudoscalar mesons | | Vector mesons | |
|---|---|---|---|
| $\pi^0$ | 135 MeV | $\rho^0$ | 775 MeV |
| $\pi^\pm$ | 140 MeV | $\rho^\pm$ | 775 MeV |
| $K^\pm$ | 494 MeV | $K^{*\pm}$ | 892 MeV |
| $K^0, \overline{K}^0$ | 498 MeV | $K^{*0}/\overline{K}^{*0}$ | 896 MeV |
| $\eta$ | 548 MeV | $\omega$ | 783 MeV |
| $\eta'$ | 958 MeV | $\phi$ | 1020 MeV |

In QED, the potential energy between two magnetic dipoles contains a term proportional to scalar product of the two dipole moments, $\boldsymbol{\mu}_i \cdot \boldsymbol{\mu}_j$. For two Dirac particles of masses $m_i$ and $m_j$, this corresponds to a potential energy term of the form

$$U \propto \frac{e}{m_i}\mathbf{S}_i \cdot \frac{e}{m_j}\mathbf{S}_j \propto \frac{\alpha}{m_i m_j}\mathbf{S}_i \cdot \mathbf{S}_j,$$

where $\alpha$ is the fine structure constant. This QED interaction term, which contributes to the hyperfine splitting of the energy levels of the hydrogen atom, is relatively small. In Chapter 10 it is shown that, apart from a numerical constant that accounts for colour, the QCD vertex has the same form as that of QED. Therefore, there will be a corresponding QCD "chromomagnetic" spin–spin interaction giving a term in the $q\overline{q}$ potential of the form

$$U \propto \frac{\alpha_S}{m_i m_j}\mathbf{S}_i \cdot \mathbf{S}_j,$$

where $\alpha_S$ is the coupling constant of QCD. Since $\alpha_S \sim 1$ is much greater than $\alpha \sim 1/137$, the chromomagnetic spin–spin interaction term is relatively large and plays an important role in determining the meson masses. For an $\ell = 0$ meson formed from a quark and an antiquark with masses $m_1$ and $m_2$, the meson mass can be written in terms of the constituent quark masses and the expectation value of the chromomagnetic spin–spin interaction

$$m(q_1 q_2) = m_1 + m_2 + \frac{A}{m_1 m_2}\langle \mathbf{S}_1 \cdot \mathbf{S}_2 \rangle, \tag{9.36}$$

where the parameter $A$ can be determined from experiment.

The scalar product $\mathbf{S}_1 \cdot \mathbf{S}_2$ in (9.36) can be obtained by writing the total spin as the vector sum, $\mathbf{S} = \mathbf{S}_1 + \mathbf{S}_2$, and squaring to give

$$\mathbf{S}^2 = \mathbf{S}_1^2 + 2\mathbf{S}_1 \cdot \mathbf{S}_2 + \mathbf{S}_2^2,$$

which implies that

$$\mathbf{S}_1 \cdot \mathbf{S}_2 = \tfrac{1}{2}\left[\mathbf{S}^2 - \mathbf{S_1}^2 - \mathbf{S_1}^2\right].$$

Therefore, the expectation value of $\mathbf{S}_1 \cdot \mathbf{S}_2$ can be written as

$$\begin{aligned}\langle \mathbf{S}_1 \cdot \mathbf{S}_2 \rangle &= \tfrac{1}{2}\left[\left\langle \mathbf{S}^2 \right\rangle - \left\langle \mathbf{S}_1^2 \right\rangle - \left\langle \mathbf{S}_2^2 \right\rangle\right] \\ &= \tfrac{1}{2}\left[s(s+1) - s_1(s_1+1) - s_2(s_2+1)\right],\end{aligned}$$

where $s_1 = s_2 = \frac{1}{2}$ and $s$ is the total spin of the $\mathrm{q\overline{q}}$ system. For the pseudoscalar mesons $s = 0$ and for the vector mesons $s = 1$ and hence (9.36) can be written

$$\text{Pseudoscalar mesons } (s=0): \quad m_P = m_1 + m_2 - \frac{3A}{4m_1 m_2}, \tag{9.37}$$

$$\text{Vector mesons } (s=1): \quad m_V = m_1 + m_2 + \frac{A}{4m_1 m_2}. \tag{9.38}$$

Hence the masses of the spin-0 pseudoscalar mesons are predicted to be lower than the masses of the spin-1 vector mesons. The observed meson masses listed in Table 9.1 are in good agreement with the predictions of the meson mass formulae of (9.37) and (9.38) with the parameters

$$m_\mathrm{d} = m_\mathrm{u} = 0.307\,\mathrm{GeV}, \quad m_\mathrm{s} = 0.490\,\mathrm{GeV} \quad \text{and} \quad A = 0.06\,\mathrm{GeV}^3. \tag{9.39}$$

The one exception is the $\eta'$, where the predicted mass of 355 MeV differs significantly from the anomalously large observed value of 958 MeV. The reason for this discrepancy is attributed to the $\eta'$ being a "flavourless" singlet state that can, in principle, mix with possible purely gluonic flavourless bound states.

### 9.6.4 The $L = 0$ uds baryons

The ground states of the (qqq) baryons are states with no orbital angular momentum in the system. Assuming SU(3) flavour symmetry, the wavefunctions for these $L = 0$ baryons can be obtained by first considering the multiplet structure for the combination of two quarks and then adding the third. This is essentially a repeat of the process used to derive the proton wavefunction in Section 9.4. Here, we will concentrate on the multiplet structure rather than the wavefunctions themselves.

Since $I_3$ and $Y$ are additive quantum numbers, the $(I_3, Y)$ values of the combination of two quarks in SU(3) are just the sums of the individual values. The multiplet structure for the combination of two quarks can be obtained by starting at one of the extreme SU(2) qq states and applying the SU(3) ladder operators. In this way it can be shown that in SU(3) flavour symmetry, the combination of two quarks leads to a symmetric sextet of states and an antisymmetric triplet of states, as shown in Figure 9.15. Since the triplet has the same $(I_3, Y)$ states as the SU(3) representation of a single antiquark, the multiplet structure arising from the combination of two quarks can be written as $3 \otimes 3 = 6 \oplus \overline{3}$.

The multiplet structure for the 27 possible flavour combinations in the qqq system is then obtained by adding a quark triplet to each of the sextet and triplet of

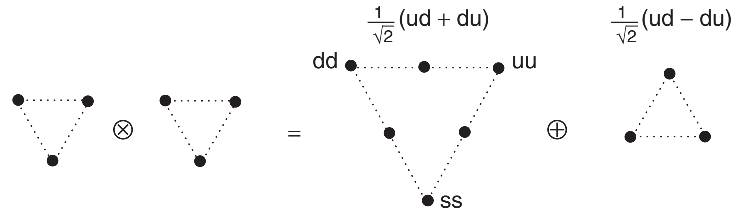


**Fig. 9.15** The multiplet structure for the combination of qq in SU(3) flavour symmetry.

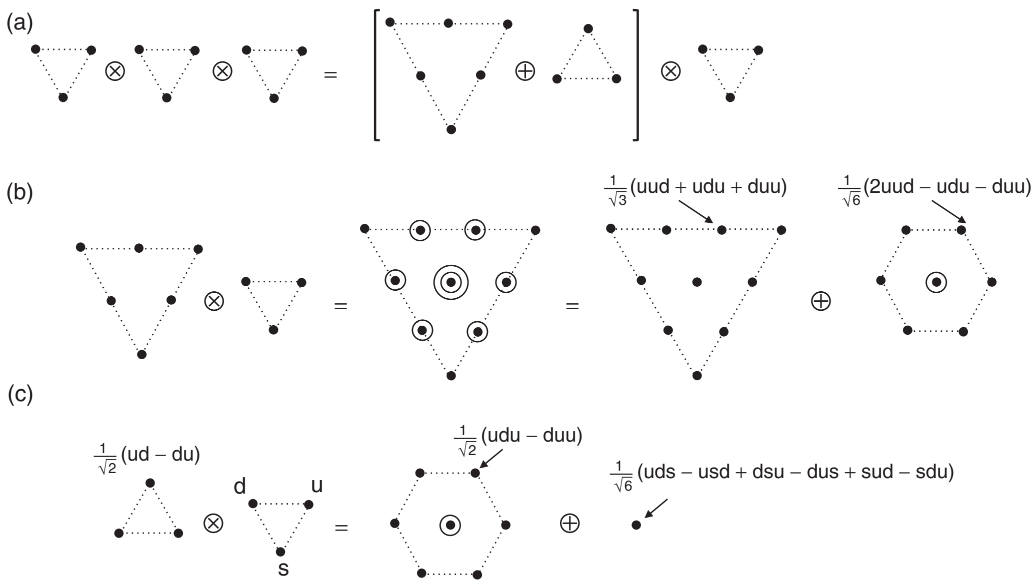


**Fig. 9.16** The $I_3$ and $Y$ assignments for the qqq multiplets in SU(3) flavour symmetry broken down into the $6 \otimes 3$ and $\bar{3} \otimes 3$ parts (shown in *b* and *c*).

Figure 9.15. In terms of the group structure, this can be written $3\otimes3\otimes3 = (6\oplus\bar{3})\otimes3$ as indicated in Figure 9.16a. Adding an additional quark to the sextet, gives a decuplet of totally symmetric states and a mixed symmetry octet, as shown in Figure 9.16b, where the states without strange quarks are exactly those identified in Section 9.4. This $10 \oplus 8$ multiplet structure can be verified by repeated application of the SU(3) ladder operators to the SU(2) states of (9.8)–(9.11) and to the states of (9.12)–(9.13) to obtain respectively the decuplet and the mixed symmetry octet.

The second set of qqq flavour states are obtained by adding a quark to the qq triplet ($\bar{3}$). In terms of the multiplet structure, this is the same as combining the SU(3) representation of a quark and antiquark ($3 \otimes \bar{3}$), giving a mixed symmetry octet and a *totally antisymmetric* singlet state, as indicated in Figure 9.16c. The wavefunctions for this octet can be obtained from the corresponding SU(2) states of (9.14)–(9.15) using the SU(3) ladder operators. Hence, 26 of the possible states

(10 + 8 + 8) can be obtained from the SU(2) qqq states using ladder operators. The final state, which must be in a singlet, is

$$|\psi_S\rangle = \frac{1}{\sqrt{6}}(\text{uds} - \text{usd} + \text{dsu} - \text{dus} + \text{sud} - \text{sdu}). \tag{9.40}$$

It is straightforward to verify that this is the singlet state by showing that the action of all the SU(3) ladder operators give zero, for example

$$\hat{T}_+|\psi_S\rangle = \frac{1}{\sqrt{6}}(\text{uus} - \text{usu} + \text{usu} - \text{uus} + \text{suu} - \text{suu}) = 0.$$

In summary, the combination of three quarks in SU(3) flavour symmetry gives a symmetric decuplet, two mixed symmetry octets and a totally antisymmetric singlet state,

$$\boxed{3 \otimes 3 \otimes 3 = 3 \otimes (6 \oplus \overline{3}) = 10 \oplus 8 \oplus 8 \oplus 1.}$$

The existence of the singlet state will have important consequences when it comes to the discussion of the SU(3) colour symmetry of QCD.

The baryon wavefunctions are obtained by combining the SU(3) flavour wavefunctions with the spin wavefunctions of Section 9.3.1, respecting the requirement that the overall baryon wavefunction has to be antisymmetric under the exchange of any two of the quarks. Since the colour wavefunction is always antisymmetric and the $\ell = 0$ spatial wavefunction is symmetric, baryon states can be formed from combinations of spin and flavour wavefunctions which are totally symmetric under the interchange of any two quarks. This can be achieved in two ways. Firstly, a symmetric spin-$\frac{3}{2}$ wavefunction can be combined with the symmetric SU(3) flavour decuplet to give ten spin-$\frac{3}{2}$ baryons (including the $\Delta$-particles). Secondly, as in (9.25), the mixed symmetry flavour octet states can be combined with the mixed symmetry spin states to give a spin-$\frac{1}{2}$ octet (including the proton and neutron). It is not possible to construct a totally symmetric *flavour* × *spin* wavefunction from the flavour singlet of (9.40) because there is no corresponding totally antisymmetric spin state formed from the combination three spin-half particles. The experimentally observed $L = 0$ baryons fit neatly into this SU(3) flavour symmetry prediction of an octet of spin-$\frac{1}{2}$ states and a decuplet of spin-$\frac{3}{2}$ states, as shown in Figure 9.17.

## Baryon masses

If the SU(3) flavour symmetry were exact, the masses of all the baryons within the octet would be the same, as would the masses of all the baryons within the decuplet. Because the strange-quark mass is greater than that of the up- and down-quarks, this is not the case. The measured masses of the $L = 0$ baryons are listed in Table 9.2. The patterns of masses within a multiplet largely reflects the number of strange quarks in the state, whereas the difference between the masses of the

**Table 9.2** Measured masses and number of strange quarks for the $L = 0$ light baryons.

| s quarks | Octet | | Decuplet | |
|---|---|---|---|---|
| 0 | p, n | 940 MeV | $\Delta$ | 1230 MeV |
| 1 | $\Sigma$ | 1190 MeV | $\Sigma^*$ | 1385 MeV |
| 1 | $\Lambda$ | 1120 MeV | | |
| 2 | $\Xi$ | 1320 MeV | $\Xi^*$ | 1533 MeV |
| 3 | | | $\Omega$ | 1670 MeV |

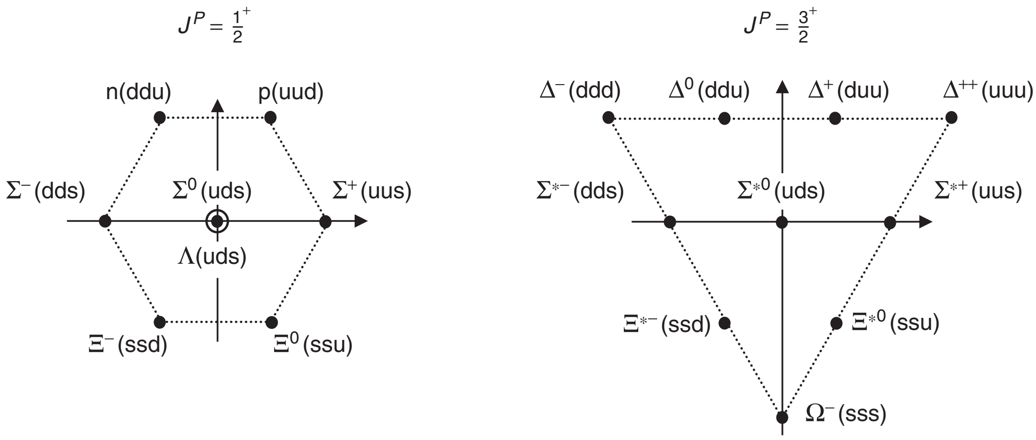


**Fig. 9.17** The observed octet and decuplet of light baryon states.

octet and decuplet states is due to the chromomagnetic spin–spin interactions of the individual quarks. Following the argument presented in Section 9.6.3, the $L = 0$ baryon mass formula is

$$m(q_1q_2q_3) = m_1 + m_2 + m_3 + A'\left(\frac{\langle \mathbf{S}_1 \cdot \mathbf{S}_2\rangle}{m_1m_2} + \frac{\langle \mathbf{S}_1 \cdot \mathbf{S}_3\rangle}{m_1m_3} + \frac{\langle \mathbf{S}_2 \cdot \mathbf{S}_3\rangle}{m_2m_3}\right), \tag{9.41}$$

where $\mathbf{S}_1$, $\mathbf{S}_2$ and $\mathbf{S}_3$ are the spin vectors of the three quarks. This expression is found to give good agreement with the observed baryon masses using

$$m_d = m_u = 0.365\,\text{GeV}, \quad m_s = 0.540\,\text{GeV} \quad \text{and} \quad A' = 0.026\,\text{GeV}^3.$$

It is important to note that the quark masses needed to explain the observed baryon masses are about 50 MeV higher than those used to describe the meson masses, as given in (9.39). Furthermore, they are very different from the fundamental up- and down-quark masses, known as the *current masses*, which are just a few MeV. The quark masses that enter the meson and baryon mass formulae are the *constituent masses*, which can be thought of as the effective masses of the quarks as they move within and interact with the QCD potential inside baryons and mesons. Since the QCD environments within baryons and mesons will be different, it should not be a surprise that the constituent masses are different for baryons and mesons. This distinction between current and constituent quark masses implies that only 1% of the mass of a proton is attributable to the masses of the quarks, the remainder arises from the energy associated with the internal QCD gluon field.

### 9.6.5 Baryon magnetic moments

In Chapter 7 it was seen that the magnetic moment of the proton differs from that expected for a point-like Dirac fermion. The experimentally measured values of the anomalous magnetic moments of the proton and neutron are $2.792\,\mu_N$ and $-1.913\,\mu_N$ respectively, where $\mu_N$ is the nuclear magneton defined as

$$\mu_N = \frac{e\hbar}{2m_\mathrm{p}}.$$

The origin of the proton and neutron anomalous magnetic moments can be explained in terms of the magnetic moments of the individual quarks and the baryon wavefunctions derived above.

Since quarks are fundamental Dirac fermions, the operators for the total magnetic moment and $z$-component of the magnetic moment are

$$\hat{\boldsymbol{\mu}} = Q\frac{e}{m}\hat{\mathbf{S}} \quad \text{and} \quad \hat{\mu}_z = Q\frac{e}{m}\hat{S}_z.$$

For spin-up ($m_s = +\frac{1}{2}$) quarks, the expectation values of the $z$-component of the magnetic moment of the up- and down-quarks are

$$\mu_\mathrm{u} = \langle \mathrm{u}\!\uparrow\! |\hat{\mu}_z|\, \mathrm{u}\!\uparrow\rangle = \left(+\tfrac{2}{3}\right)\frac{e\hbar}{2m_\mathrm{u}} = +\frac{2m_\mathrm{p}}{3m_\mathrm{u}}\mu_N, \tag{9.42}$$

$$\mu_\mathrm{d} = \langle \mathrm{d}\!\uparrow\! |\hat{\mu}_z|\, \mathrm{d}\!\uparrow\rangle = \left(-\tfrac{1}{3}\right)\frac{e\hbar}{2m_\mathrm{d}} = -\frac{m_\mathrm{p}}{3m_\mathrm{d}}\mu_N. \tag{9.43}$$

The corresponding expressions for the spin-down states are

$$\langle \mathrm{u}\!\downarrow\! |\hat{\mu}_z|\, \mathrm{u}\!\downarrow\rangle = -\mu_\mathrm{u} \quad \text{and} \quad \langle \mathrm{d}\!\downarrow\! |\hat{\mu}_z|\, \mathrm{d}\!\downarrow\rangle = -\mu_\mathrm{d}.$$

The total magnetic moment of a baryon is the vector sum of the magnetic moments of the three constituent quarks

$$\hat{\boldsymbol{\mu}} = \hat{\boldsymbol{\mu}}^{(1)} + \hat{\boldsymbol{\mu}}^{(2)} + \hat{\boldsymbol{\mu}}^{(3)},$$

where $\hat{\boldsymbol{\mu}}^{(i)}$ is the magnetic moment operator which acts on the $i$th quark. Therefore, the magnetic moment of the proton can be written

$$\mu_\mathrm{p} = \langle\hat{\mu}_z\rangle = \langle \mathrm{p}\!\uparrow\! |\hat{\mu}_z^{(1)} + \hat{\mu}_z^{(2)} + \hat{\mu}_z^{(3)}\, |\mathrm{p}\!\uparrow\rangle. \tag{9.44}$$

The order that the quarks appear in the proton wavefunction does not affect the calculation of the magnetic moment and it is sufficient to write

$$|\mathrm{p}\uparrow\rangle = \tfrac{1}{\sqrt{6}}(2\mathrm{u}\uparrow\mathrm{u}\uparrow\mathrm{d}\downarrow - \mathrm{u}\uparrow\mathrm{u}\downarrow\mathrm{d}\uparrow - \mathrm{u}\downarrow\mathrm{u}\uparrow\mathrm{d}\uparrow),$$

and thus (9.44) can be written as

$$\mu_\mathrm{p} = \tfrac{1}{6}\,\langle(2\mathrm{u}\uparrow\mathrm{u}\uparrow\mathrm{d}\downarrow - \mathrm{u}\uparrow\mathrm{u}\downarrow\mathrm{d}\uparrow - \mathrm{u}\downarrow\mathrm{u}\uparrow\mathrm{d}\uparrow)\,|\hat{\mu}_z|\,(2\mathrm{u}\uparrow\mathrm{u}\uparrow\mathrm{d}\downarrow - \mathrm{u}\uparrow\mathrm{u}\downarrow\mathrm{d}\uparrow - \mathrm{u}\downarrow\mathrm{u}\uparrow\mathrm{d}\uparrow)\rangle\,,$$

where $\hat{\mu}_z = \hat{\mu}_z^{(1)} + \hat{\mu}_z^{(2)} + \hat{\mu}_z^{(3)}$. Because of the orthogonality of the quark flavour and spin states, for example $\langle \mathrm{u}\uparrow\mathrm{u}\uparrow\mathrm{d}\downarrow\,|\,\mathrm{u}\downarrow\mathrm{u}\uparrow\mathrm{d}\uparrow\rangle = 0$, the expression for the proton magnetic moment reduces to

$$\begin{aligned}\mu_\mathrm{p} = {} & \tfrac{4}{6}\,\langle \mathrm{u}\uparrow\mathrm{u}\uparrow\mathrm{d}\downarrow|\hat{\mu}_z|\,\mathrm{u}\uparrow\mathrm{u}\uparrow\mathrm{d}\downarrow\rangle + \tfrac{1}{6}\,\langle \mathrm{u}\uparrow\mathrm{u}\downarrow\mathrm{d}\uparrow|\hat{\mu}_z|\,\mathrm{u}\uparrow\mathrm{u}\downarrow\mathrm{d}\uparrow\rangle \\ & + \tfrac{1}{6}\,\langle \mathrm{u}\downarrow\mathrm{u}\uparrow\mathrm{d}\uparrow|\hat{\mu}_z|\,\mathrm{u}\downarrow\mathrm{u}\uparrow\mathrm{d}\uparrow\rangle\,.\end{aligned} \tag{9.45}$$

Equation (9.45) can be evaluated using

$$\begin{aligned}\hat{\mu}_z|\mathrm{u}\uparrow\rangle &= +\mu_\mathrm{u}\,|\mathrm{u}\uparrow\rangle \quad \text{and} \quad \hat{\mu}_z|\mathrm{u}\downarrow\rangle = -\mu_\mathrm{u}\,|\mathrm{u}\downarrow\rangle,\\ \hat{\mu}_z|\mathrm{d}\uparrow\rangle &= +\mu_\mathrm{d}\,|\mathrm{d}\uparrow\rangle \quad \text{and} \quad \hat{\mu}_z|\mathrm{d}\downarrow\rangle = -\mu_\mathrm{d}\,|\mathrm{d}\downarrow\rangle,\end{aligned}$$

giving

$$\mu_\mathrm{p} = \tfrac{4}{6}\,(\mu_\mathrm{u} + \mu_\mathrm{u} - \mu_\mathrm{d}) + \tfrac{1}{6}\,(\mu_\mathrm{u} - \mu_\mathrm{u} + \mu_\mathrm{d}) + \tfrac{1}{6}\,(-\mu_\mathrm{u} + \mu_\mathrm{u} + \mu_\mathrm{d})\,.$$

Therefore, the quark model prediction for the magnetic moment of the proton is

$$\mu_\mathrm{p} = \tfrac{4}{3}\mu_\mathrm{u} - \tfrac{1}{3}\mu_\mathrm{d}.$$

The prediction for the magnetic moment of the neutron can be written down by replacing $\mathrm{u} \rightarrow \mathrm{d}$ and vice versa,

$$\mu_\mathrm{n} = \tfrac{4}{3}\mu_\mathrm{d} - \tfrac{1}{3}\mu_\mathrm{u}.$$

Assuming that $m_\mathrm{u} \approx m_\mathrm{d}$, the relations of (9.42) and (9.43) imply that $\mu_\mathrm{u} = -2\mu_\mathrm{d}$. Consequently, the ratio of the proton and neutron magnetic moments is predicted to be

$$\frac{\mu_\mathrm{p}}{\mu_\mathrm{n}} = \frac{4\mu_\mathrm{u} - \mu_\mathrm{d}}{4\mu_\mathrm{d} - \mu_\mathrm{u}} = -\frac{3}{2},$$

which is in reasonable agreement with the experimentally measured value of $-1.46$. The best agreement between the quark model predictions and the measured values of the magnetic moments of the $L = 0$ baryons is obtained with

$$m_\mathrm{u} = 0.338\,\mathrm{GeV}, \quad m_\mathrm{d} = 0.322\,\mathrm{GeV} \quad \text{and} \quad m_\mathrm{s} = 0.510\,\mathrm{GeV}.$$

Using these values in (9.42) and (9.43) gives $\mu_\mathrm{u} = +1.85\mu_N$ and $\mu_\mathrm{d} = -0.97\mu_N$, reproducing the observed values of the proton and neutron magnetic moments.

### 9.6.6 Final words on SU(3) flavour symmetry

Whilst the SU(3) flavour symmetry is only approximate, it is able to account for the observed states of the $L = 0$ mesons and baryons. Furthermore, the hadron wavefunctions derived in the context of SU(3) flavour symmetry can be used to obtain reasonable predictions for baryon and meson masses and the baryon magnetic moments. If anything, it is perhaps surprising that the predictions from SU(3) flavour symmetry give such reasonable results. After all, the SU(3) flavour symmetry can be only approximate because the mass of the strange quark is about 0.1 GeV greater than the masses of the up- and down-quarks, although this mass difference is relatively small compared to the typical QCD binding energy which is of order 1 GeV. A further issue with the static quark model is that the hadronic states have been treated as bound states of valence quarks, whereas from the discussion of deep inelastic scattering it is clear that hadrons are far more complex. To some extent, these additional degrees of freedom are accounted for in the *constituent masses* of the quarks used to obtain the predictions for meson and baryon masses and the baryon magnetic moments. These masses are much larger than the current masses listed in Table 1.1; most of the mass of the hadrons originates from of the energy of the strongly interacting sea of virtual quarks and gluons.

The above discussion was restricted to the approximate SU(3) flavour symmetry of the three light quarks. It is tempting to extend this treatment to an SU(4) flavour symmetry including the charm quark. However, this makes little sense; the difference between the mass of the charm quark and the light quarks is greater than 1 GeV, which is the typical QCD binding energies of hadrons. For this reason, the Hamiltonian for the hadronic states does not possess even an approximate SU(4) flavour symmetry.

## Summary

In this chapter a number of important concepts were introduced. Symmetries of the Hamiltonian were associated with unitary transformations expressed in terms of Hermitian generators

$$\hat{U}(\alpha) = \exp(i\alpha \cdot \hat{G}).$$

In this way, each symmetry of the Hamiltonian is associated with an observable conserved quantity.

The flavour symmetry of the static quark model was used to illustrate these ideas and to introduce the SU(2) and SU(3) groups. Based on symmetry arguments alone, it was possible to derive static wavefunctions for the mesons and baryons formed from u, d and s quarks. The static quark model was shown to provide a good description of the masses and magnetic moments of the light hadrons. In the following chapter, these ideas will be extended to the abstract SU(3) local gauge symmetry that lies at the heart of QCD.

## 9.7 *Addendum: Flavour symmetry revisited

In the derivation of the proton wavefunction, given in Section 9.4, the overall wavefunction,

$$\psi = \phi_{\text{flavour}}\,\chi_{\text{spin}}\,\xi_{\text{colour}}\,\eta_{\text{space}},$$

was required to be antisymmetric. For cases where the flavour wavefunction describes like particles, for example $\phi_{\text{flavour}} = \text{uuu}$, the requirement of an overall antisymmetric wavefunction is just an expression of Pauli exclusion principle, which arises from the spin-statistics of fermions. It is less obvious why this should also apply to the more general case with different quark flavours; the reasoning is subtle.

In quantum field theory an up-quark state with spin $r$ is expressed by the action of the creation operator $a^{\dagger}_{+r}$ on the vacuum state,

$$|\text{u}\uparrow\rangle = a^{\dagger}_{+r}|0\rangle,$$

where the + sign refers to the creation of the $I_3 = +\frac{1}{2}$ state labelling an up-quark in SU(2) flavour symmetry. The creation operator $a^{\dagger}_{+r}$ satisfies the requirements of fermion spin statistics, which can be written as the anticommutator

$$\{a^{\dagger}_{+r}, a^{\dagger}_{+r}\} = 0,$$

which implies

$$a^{\dagger}_{+r}a^{\dagger}_{+r}|0\rangle = 0, \tag{9.46}$$

and therefore two identical particles can not be produced in the same state. For the SU(2) isospin flavour symmetry $\hat{T}_-|\text{u}\uparrow\rangle = |\text{d}\uparrow\rangle$, which implies

$$\hat{T}_- a^{\dagger}_{+r}|0\rangle = a^{\dagger}_{-r}|0\rangle,$$

where $a^{\dagger}_{-r}$ is the creation operator for a spin-up down quark with $I_3 = -\frac{1}{2}$. Therefore one can write $\hat{T}_- a^{\dagger}_{+r} = a^{\dagger}_{-r}$. Applying the isospin lowering operator to (9.46) gives

$$\hat{T}_-(a^{\dagger}_{+r}a^{\dagger}_{+r})|0\rangle = a^{\dagger}_{-r}a^{\dagger}_{+r}|0\rangle + a^{\dagger}_{+r}a^{\dagger}_{-r}|0\rangle = 0,$$

and hence

$$\{a^{\dagger}_{+r}, a^{\dagger}_{-r}\} = 0.$$

Therefore, within the assumed SU(2) flavour symmetry, the creation operators for up- and down-quarks satisfy the same anticommutation relations as the creation operators for two up-quarks or two down-quarks. Consequently, within the SU(2) or SU(3) flavour symmetries, the requirement that the overall wavefunction is anti-symmetric applies equally to states where the flavours of the quarks are different.

## Problems

**9.1** By writing down the general term in the binomial expansion of

$$\left(1 + i\frac{1}{n}\boldsymbol{\alpha}\cdot\hat{\mathbf{G}}\right)^n,$$

show that

$$\hat{U}(\boldsymbol{\alpha}) = \lim_{n\to\infty}\left(1 + i\frac{1}{n}\boldsymbol{\alpha}\cdot\hat{\mathbf{G}}\right)^n = \exp\left(i\boldsymbol{\alpha}\cdot\mathbf{G}\right).$$

**9.2** For an infinitesimal rotation about the $z$-axis through an angle $\epsilon$ show that

$$\hat{U} = 1 - i\epsilon\hat{J}_z,$$

where $\hat{J}_z$ is the angular momentum operator $\hat{J}_z = x\hat{p}_y - y\hat{p}_x$.

**9.3** By considering the isospin states, show that the rates for the following strong interaction decays occur in the ratios

$$\Gamma(\Delta^- \to \pi^-\mathrm{n}) : \Gamma(\Delta^0 \to \pi^-\mathrm{p}) : \Gamma(\Delta^0 \to \pi^0\mathrm{n}) : \Gamma(\Delta^+ \to \pi^+\mathrm{n}) :$$
$$\Gamma(\Delta^+ \to \pi^0\mathrm{p}) : \Gamma(\Delta^{++} \to \pi^+\mathrm{p}) = 3 : 1 : 2 : 1 : 2 : 3.$$

**9.4** If quarks and antiquarks were spin-zero particles, what would be the multiplicity of the $L = 0$ multiplet(s). Remember that the overall wavefunction for bosons must be symmetric under particle exchange.

**9.5** The neutral vector mesons can decay leptonically through a virtual photon, for example by $V(\mathrm{q}\overline{\mathrm{q}}) \to \gamma \to \mathrm{e}^+\mathrm{e}^-$. The matrix element for this decay is proportional to $\langle\psi|\hat{Q}_\mathrm{q}|\psi\rangle$, where $\psi$ is the meson flavour wavefunction and $\hat{Q}_\mathrm{q}$ is an operator that is proportional to the quark charge. Neglecting the relatively small differences in phase space, show that

$$\Gamma(\rho^0 \to \mathrm{e}^+\mathrm{e}^-) : \Gamma(\omega \to \mathrm{e}^+\mathrm{e}^-) : \Gamma(\phi \to \mathrm{e}^+\mathrm{e}^-) \approx 9 : 1 : 2.$$

**9.6** Using the meson mass formulae of (9.37) and (9.38), obtain predictions for the masses of the $\pi^\pm$, $\pi^0$, $\eta$, $\eta'$, $\rho^0$, $\rho^\pm$, $\omega$ and $\phi$. Compare the values obtained to the experimental values listed in Table 9.1.

**9.7** Compare the experimentally measured values of the masses of the $J^P = \frac{3}{2}^+$ baryons, given in Table 9.2, with the predictions of (9.41). You will need to consider the combined spin of *any* two quarks in a spin-$\frac{3}{2}$ baryon state.